Charged particle motion in a time-dependent flux-driven ring: an exactly solvable model

2007 J. Phys.: Condens. Matter 19 176224

(http://iopscience.iop.org/0953-8984/19/17/176224)

IOP PUBLISHING JOURNAL OF PHYSICS: CONDENSED MATTER

J. Phys.: Condens. Matter **19** (2007) 176224 (10pp) doi:10.1088/0953-8984/19/17/176224

Charged particle motion in a time-dependent flux-driven ring: an exactly solvable model

Pi-Gang Luan[1] **and Chi-Shung Tang**[2]

[1] Department of Optics and Photonics, National Central University, Chung-Li 32054, Taiwan
[2] Research Center for Applied Sciences, Academia Sinica, Taipei 11529, Taiwan

Received 31 October 2006, in final form 5 March 2007
Published 10 April 2007
Online at stacks.iop.org/JPhysCM/19/176224

Abstract
We consider a charged particle driven by a time-dependent flux threading a quantum ring. The dynamics of the charged particle is investigated using a classical treatment, a Fourier expansion technique, a time-evolution method, and the Lewis–Riesenfeld approach. We have shown that, by properly managing the boundary conditions, a time-dependent wavefunction can be obtained using a general non-Hermitian time-dependent invariant, which is a specific linear combination of initial angular-momentum and azimuthal-angle operators. It is shown that the linear invariant eigenfunction can be realized as a Gaussian-type wavepacket with a peak moving along the classical angular trajectory, while the distribution of the wavepacket is determined by the ratio of the coefficient of the initial angle to that of the initial canonical angular momentum. From the topologically nontrivial nature as well as the classical trajectory and angular momentum, one can determine the dynamical motion of the wavepacket. It should be noted that the peak position is no longer an expectation value of the angle operator, and hence the Ehrenfest theorem is not directly applicable in such a topologically nontrivial system.

1. Introduction

A charged particle driven by a time-dependent perturbation in a quantum system is a nontrivial fundamental issue [1–8]. One can access the charged particle wavefunction by placing it in a quantum ring threaded by a time-dependent magnetic flux. The vector potential $\mathbf{A}(t)$ associated with the time-dependent flux $\Phi(t)$ times the charge q leads to a phase shift proportional to the number of flux quanta penetrating the ring; this is known as Aharonov–Bohm (AB) effect [9–11]. In adiabatic cyclic evolution, Berry [12] was the first to discover that there exists a geometric phase. Later on, Aharonov and Anandan (AA) removed the adiabatic restriction to explore the geometric phase for any cyclic evolution [13]. Time-dependent fields are also used to deal with field-driven Zener tunnelling, in which nonadiabaticity plays a crucial role [14–16].

In mesoscopic systems, a number of manifestations of the AB effect have been predicted and verified [17–22]. On the other hand, Stern demonstrated that the Berry phase affects the particle motion in the ring similarly to the AB effect, and a time-dependent Berry phase induces a motive force [24]. It was found experimentally [23] that a quantum ring threaded by a static magnetic field displays persistent currents oscillating in period of $\Phi_0 = h/q$, the ratio of Planck constant and charge of a particle.

In the present work, we consider a noninteracting spinless charged particle moving cyclically in a quantum ring in the presence of a time-dependent vector potential. Such a particle motion can be described by the time-dependent Schrödinger equation

$$i\hbar \frac{\partial \psi}{\partial t} = \hat{H}(t)\psi, \tag{1}$$

where the Hamiltonian $\hat{H}(t)$ is induced by an external time-dependent vector potential $\mathbf{A}(t)$, given by

$$\begin{aligned} \hat{H}(t) &= \frac{1}{2m}\left[\hat{\mathbf{P}} - q\mathbf{A}(t)\right]^2 \\ &= \frac{1}{2I}\left[\hat{L} - qRA(t)\right]^2. \end{aligned} \tag{2}$$

Here $\hat{\mathbf{P}} = \mathbf{e}_\theta \hat{P}_\theta$ is the canonical momentum operator with $\mathbf{e}_\theta$ being the unit vector along the azimuthal angle θ; $\hat{L} = \hat{L}_z = (\hat{\mathbf{r}} \times \hat{\mathbf{P}})_z$ is the canonical angular momentum operator in the z direction; $I = mR^2$ is the moment of inertia of the particle; $\mathbf{A}(t) = A(t)\mathbf{e}_\theta$ is the vector potential; and R is the radius of the circular ring. This time-dependent dynamical problem can be solved by taking into account the Fourier expansion, time evolution operator, and Lewis–Riesenfeld (LR) method [25, 26].

2. A classical treatment

We first analyse the time-dependent problem in a classical manner. The time-varying magnetic flux induces an electric field $\mathbf{E} = E\mathbf{e}_\theta$ such that $E = -\partial A/\partial t$. The charged particle thus obtain a kinematic momentum increment during the time interval from 0 to t, namely

$$\Delta p_c = \Delta(mv) = m[v(t) - v(0)] = -q\,[A(t) - A(0)], \tag{3}$$

where $p_c = mv$ is the kinematic momentum. It should be noted that both p_c and qA are not conservative quantities, while from equation (3) we see that the canonical momentum P_c is a constant of motion:

$$P_c(t) = mv(t) + qA(t) = mv(0) + qA(0) = P_c(0). \tag{4}$$

Comparing the two identities in equation (2), we see that the result of equation (3) is equivalent to

$$\Delta l_c = I\,[\omega(t) - \omega(0)] = -\frac{q}{2\pi}\,[\Phi(t) - \Phi(0)], \tag{5}$$

where $l_c = (\mathbf{r} \times \mathbf{p}_c)_z = I\omega$ indicates the kinematic angular momentum, ω is the angular velocity, Φ is the magnetic flux threading the ring, and the fact that

$$\Phi(t) = 2\pi\, RA(t) \tag{6}$$

has been used. Also, equation (4) leads us to obtain the following relations:

$$L_c(t) = l_c(t) + \frac{q}{2\pi}\Phi(t) = l_c(0) + \frac{q}{2\pi}\Phi(0) = L_c(0). \tag{7}$$

These identities imply that the *canonical angular momentum* L_c, defined by $(\mathbf{r} \times \mathbf{P}_c)_z$, is also a constant of motion.

Now we define the *writhing number* as

$$n_\Phi(t) \equiv \frac{\Phi(t)}{\Phi_0}, \tag{8}$$

where $\Phi_0 = h/q$ is a flux quantum. We also define

$$L_c \equiv n_0\hbar, \qquad l_c(t) \equiv n_c(t)\hbar, \tag{9}$$

then we have

$$n_c(t) = n_0 - n_\Phi(t). \tag{10}$$

All of these ns are *real* numbers.

Now the angular position of the driven particle is given by

$$\begin{aligned}\theta_c(t) &= \theta_0 + \int_0^t \frac{n_c(\tau)\hbar}{I}\,d\tau \\ &= \theta_0 + \omega_0 t - \int_0^t \frac{n_\Phi(\tau)\hbar}{I}\,d\tau.\end{aligned} \tag{11}$$

Here θ_0 indicates the initial azimuthal angle; and $\omega_0 = \omega(0) = n_0\hbar/I$ stands for the initial angular velocity. Below we denote the initial kinematic angular momentum $l_0 \equiv l_c(0)$ for simplicity.

Hereafter we solve the quantum version of the problem, i.e., equation (1), using three different methods. The classical quantities $l_c(t)$ and $\theta_c(t)$ will also appear in the expressions of the wavefunction. Their roles in the quantum problem will be further explored.

3. A Fourier expansion method

The simplest method for solving the time-dependent flux-driven problem is the Fourier expansion method. The first thing about the system we discuss is that the wavefunction satisfies the periodic boundary condition:

$$\psi(\theta, t) = \psi(\theta + 2\pi, t). \tag{12}$$

The most general form of ψ for the present problem is thus written as

$$\psi(\theta, t) = \sum_{n=-\infty}^{\infty} c_n f_n(t) e^{in\theta}, \tag{13}$$

where the c_ns are appropriate coefficients to be determined by the initial and the boundary conditions.

Substituting equation (13) into (1), we can find the identity

$$\sum_{n=-\infty}^{\infty} i\hbar\, c_n \dot{f}_n(t) e^{in\theta} = \sum_{n=-\infty}^{\infty} c_n f_n(t) \frac{(n\hbar - qRA(t))^2}{2I} e^{in\theta}. \tag{14}$$

Solving equation (14), after some procedures we obtain

$$f_n(t) = \exp\left\{-\frac{i}{2I\hbar}\int_0^t \left[n\hbar - qRA(t)\right]^2 dt\right\}, \tag{15}$$

and thus

$$\psi(\theta, t) = \sum_{n=-\infty}^{\infty} c_n \exp\left\{-\frac{i\hbar}{2I}\int_0^t [n - n_\Phi(\tau)]^2\, d\tau + in\theta\right\}. \tag{16}$$

As a simple example, let us choose

$$c_n = N \exp\left[-\sigma^2(n - n_0)^2 - i\theta_0(n - n_0)\right], \tag{17}$$

where N indicates an appropriate normalization constant; σ, θ_0 and n_0 are real numbers, and n is an integer.

Substituting equation (17) into (16), we get

$$\psi(\theta,t) = N \exp\left[-\frac{\mathrm{i}}{\hbar}\int_0^t \frac{l_{\mathrm{c}}^2(\tau)}{2I}\,\mathrm{d}\tau + \mathrm{i}n_0\theta_{\mathrm{c}}(t)\right] \times \sum_{n=-\infty}^{\infty} \exp\left\{-\sigma^2\left(1+\frac{\mathrm{i}t}{T}\right)(n-n_0)^2 + \mathrm{i}n\,[\theta-\theta_{\mathrm{c}}(t)]\right\}, \tag{18}$$

where $T = 2I\sigma^2/\hbar$. Applying the Poisson summation formula

$$\sum_{n=-\infty}^{\infty} f(n) = \sum_{n=-\infty}^{\infty}\left(\int_{-\infty}^{\infty} f(x)\mathrm{e}^{\mathrm{i}2\pi nx}\,\mathrm{d}x\right) \tag{19}$$

on the function

$$f(x) = \exp\left[-\sigma^2\left(1+\frac{\mathrm{i}t}{T}\right)(x-n_0)^2 + \mathrm{i}\,(\theta-\theta_{\mathrm{c}}(t))\,x\right], \tag{20}$$

we can obtain an alternative expression

$$\psi(\theta,t) = N\sqrt{\frac{\pi}{\sigma^2(1+\frac{\mathrm{i}t}{T})}}\exp\left[-\frac{\mathrm{i}}{\hbar}\int_0^t \frac{l_{\mathrm{c}}^2(\tau)}{2I}\,\mathrm{d}\tau\right] \times \sum_{n=-\infty}^{\infty}\exp\left[-\frac{(\theta-\theta_{\mathrm{c}}(t)+2n\pi)^2}{4\sigma^2\left(1+\frac{\mathrm{i}t}{T}\right)} + \mathrm{i}n_0(\theta+2n\pi)\right]. \tag{21}$$

We note that equation (16) is the general solution of the problem, whereas equations (18) and (21) are two different expressions for a special solution defined by the c_n coefficients of equation (17). When $\sigma^2 t/T < 1$, it should be noted that equation (18) converges slowly, while equation (21) converges quickly. This means that in the short-time limit, $t < T/\sigma^2$, the wavefunction is better described by a circulating wavepacket. However, for the case of a long-time limit, $t \gg T/\sigma^2$, equation (18) has fast convergency; this is because, in this expression, only the $n \approx n_0$ terms are important. If we further assume n_0 to be an integer, then at large t the wavefunction approaches a circulating plane wave that is characterized by n_0, namely

$$\psi(\theta,t) \approx N\exp\left[-\frac{\mathrm{i}}{\hbar}\int_0^t \frac{l_{\mathrm{c}}^2(\tau)}{2I}\,\mathrm{d}\tau + \mathrm{i}n_0\theta\right].$$

From these findings we conclude that equation (21) describes the short-time behaviour and equation (18) describes the long-time behaviour of the ring system when the wavefunction is defined by equation (17).

4. A time evolution method

In this section, we shall present how to get the general solution shown in the previous section in terms of the time evolution operator $\hat{U}(t)$. The state $|\psi(t)\rangle$ is connected with the initial state $|\psi(0)\rangle$ through

$$|\psi(t)\rangle = \hat{U}(t)|\psi(0)\rangle \tag{22}$$

and the wavefunction $\psi(\theta,t)$ is given by

$$\psi(\theta,t) = \langle\theta|\hat{U}(t)|\psi(0)\rangle, \tag{23}$$

where $|\theta\rangle$ is the θ-eigenket in the Schrödinger picture that will be explained later.

To begin with, we introduce the canonical commutator

$$[\hat{\theta}(0), \hat{L}(0)] = \mathrm{i}\hbar. \tag{24}$$

From this identity, we have

$$[\hat{\theta}(t), \hat{L}(t)] = \hat{U}^\dagger(t)[\hat{\theta}(0), \hat{L}(0)]\hat{U}(t) = \mathrm{i}\hbar. \tag{25}$$

Utilizing equation (25) we can derive

$$\frac{\mathrm{d}\hat{L}(t)}{\mathrm{d}t} = \frac{[\hat{L}(t), \hat{H}(t)]}{\mathrm{i}\hbar} = 0, \tag{26}$$

and we thus obtain the identity $\hat{L}(t) = \hat{L}(0)$. Following a similar procedure it is easy to obtain

$$\frac{\mathrm{d}\hat{\theta}(t)}{\mathrm{d}t} = \frac{[\hat{\theta}(t), \hat{H}(t)]}{\mathrm{i}\hbar} = \frac{\hat{L}(0) - n_\Phi(t)\hbar}{I}, \tag{27}$$

which gives us

$$\hat{\theta}(t) = \hat{\theta}(0) + \frac{\hat{L}(0)t}{I} - \int_0^t \frac{n_\Phi(\tau)\hbar}{I}\,\mathrm{d}\tau. \tag{28}$$

Here we see that the canonical angular momentum is a constant of motion. This is consistent with the classical results discussed in section 2.

From the above results we have

$$[\hat{H}(t), \hat{H}(t')] = 0 \tag{29}$$

for any two times t and t'. Hence the time evolution operator is simply given by

$$\begin{aligned}\hat{U}(t) &= \exp\left[-\frac{\mathrm{i}}{\hbar}\int_0^t \hat{H}(\tau)\,\mathrm{d}\tau\right] \\ &= \exp\left[-\frac{\mathrm{i}\hbar}{2I}\int_0^t \left(\frac{\hat{L}(0)}{\hbar} - n_\Phi(\tau)\right)^2 \mathrm{d}\tau\right].\end{aligned} \tag{30}$$

To proceed further, we define $|n\rangle$ as the eigenket of $\hat{L}(0)$ obeying

$$\hat{L}(0)|n\rangle = n\hbar|n\rangle. \tag{31}$$

Then we assume that $|\theta\rangle$ is an eigenket of $\mathrm{e}^{\mathrm{i}\hat{\theta}(0)}$ obeying

$$\mathrm{e}^{\mathrm{i}\hat{\theta}(0)}|\theta\rangle = \mathrm{e}^{\mathrm{i}\theta}|\theta\rangle. \tag{32}$$

The orthogonal conditions of the two eigenkets can thus be expressed by

$$\langle m|n\rangle = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{e}^{\mathrm{i}(m-n)\theta}\,\mathrm{d}\theta = \delta_{mn} \tag{33}$$

and

$$\langle\theta|\theta'\rangle = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} \mathrm{e}^{\mathrm{i}n(\theta-\theta')} = \delta(\theta-\theta'). \tag{34}$$

These two orthogonal conditions can be derived from the closure relations

$$\sum_{n=-\infty}^{\infty} |n\rangle\langle n| = 1, \qquad \int_0^{2\pi} \mathrm{d}\theta\,|\theta\rangle\langle\theta| = 1 \tag{35}$$

and taking into account the definition

$$\langle\theta|n\rangle = \frac{1}{\sqrt{2\pi}}\mathrm{e}^{\mathrm{i}n\theta} = \langle n|\theta\rangle^*. \tag{36}$$

It should be noted that both θ and θ' are defined in the interval $[0, 2\pi)$. In the coordinate representation, $e^{in\theta}$ is an eigenfunction of $\hat{L}_{rep} = -i\hbar\partial/\partial\theta$ with corresponding eigenvalue $n\hbar$. This result can be expressed as

$$\langle\theta|\hat{L}(0)|n\rangle = \hat{L}_{rep}\langle\theta|n\rangle = n\hbar\langle\theta|n\rangle. \tag{37}$$

The wavefunction ψ now can be calculated:

$$\begin{aligned}\psi(\theta,t) &= \sum_{n=-\infty}^{\infty}\langle\theta|\hat{U}(t)|n\rangle\langle n|\psi(0)\rangle \\ &= \sum_{n=-\infty}^{\infty}\frac{\langle n|\psi(0)\rangle}{\sqrt{2\pi}}e^{-\frac{i\hbar}{2I}\int_0^t[n-n_\Phi(\tau)]^2\,d\tau+in\theta}.\end{aligned} \tag{38}$$

If we define

$$c_n = \frac{\langle n|\psi(0)\rangle}{\sqrt{2\pi}}, \tag{39}$$

then the result of equation (38) becomes that of equation (16).

Using the time evolution operator $\hat{U}(t)$, we have indeed found the general solution of equation (16). Based on the commutativity of the Hamiltonian operator at different times (equation (29)), the $\hat{U}(t)$ operator can be constructed straightforwardly by simple integration.

5. The Lewis–Riesenfeld method

In this section, we briefly review the LR method and then apply it to solve the present problem. We shall show that the LR method is not directly applicable; however, a simple modification concerning the boundary condition makes it applicable to solving problems with periodic boundary conditions.

Traditionally, to utilize the LR method [26] of solving a time-dependent system, we have to find an operator $\hat{Q}(t)$ such that

$$i\hbar\frac{d\hat{Q}}{dt} = i\hbar\frac{\partial\hat{Q}}{\partial t} + [\hat{Q},\hat{H}] = 0, \tag{40}$$

and then find its eigenfunction $\varphi_\lambda(\theta,t)$ satisfying

$$\hat{Q}(t)\,\varphi_\lambda(\theta,t) = \lambda\,\varphi_\lambda(\theta,t), \tag{41}$$

with λ being the corresponding eigenvalue. A wavefunction $\psi_\lambda(\theta,t)$ satisfying equation (1) is then obtained via the relation

$$\psi_\lambda(\theta,t) = e^{i\alpha_\lambda(t)}\,\varphi_\lambda(\theta,t), \tag{42}$$

where $\alpha(t)$ is a function of time only, satisfying

$$\dot{\alpha}_\lambda = \varphi_\lambda^{-1}(i\partial/\partial t - \hat{H}/\hbar)\varphi_\lambda. \tag{43}$$

A general solution ψ of equation (1) is then given by

$$\psi(\theta,t) = \sum_\lambda g(\lambda)\psi_\lambda(\theta,t), \tag{44}$$

where $g(\lambda)$ is a weight function for λ.

To proceed, let us assume that the time-dependent invariant operator $\hat{Q}(t)$ takes the linear form [5, 6]

$$\hat{Q}(t) = a(t)\hat{L} + b(t)\hat{\theta} + c(t), \tag{45}$$

in which $a(t)$, $b(t)$, and $c(t)$ are time-dependent c-number functions to be determined.

Substituting equation (45) into equation (40) and solving these operator equations, we get

$$a(t) = a_0 - \frac{b_0 t}{I}, \qquad b(t) = b_0, \tag{46}$$

$$c(t) = c_0 + b_0 \int_0^t \frac{n_\Phi(\tau)\hbar}{I}\,\mathrm{d}\tau, \tag{47}$$

where a_0, b_0, and c_0 are arbitrary complex constants. Furthermore, substituting equations (46) and (47) into (45), we find

$$\hat{Q}(t) = a_0\,\hat{L}(0) + b_0\,\hat{\theta}(0) + c_0 = \hat{Q}(0). \tag{48}$$

In other words, the invariant $\hat{Q}$ in the Heisenberg picture is precisely the linear combination of the initial canonical angular momentum $\hat{L}(0)$ and the initial azimuthal angle $\hat{\theta}(0)$ with an arbitrary constant c_0. Note that in our system the $\hat{L}$ operator is also an invariant.

It is interesting to ask how the eigenvalue λ evolves in time. Multiplying the factor $\mathrm{e}^{\mathrm{i}\alpha(t)}$ on both sides of equation (41), we get

$$\hat{Q}(t)\,\psi_\lambda(\theta, t) = \lambda\,\psi_\lambda(\theta, t). \tag{49}$$

Partially differentiating both sides of equation (49) with respect to time and using equation (40), we find

$$\lambda(t) = \lambda(0); \tag{50}$$

thus λ is a constant.

To find a solution of equation (1), we have to solve equation (41) first. By solving equation (41), we get

$$\varphi_\lambda(\theta, t) = \exp\left[\frac{\mathrm{i}}{\hbar}\left(\mu(t)\theta - \frac{1}{2}\nu(t)\theta^2\right)\right], \tag{51}$$

where

$$\mu(t) = \frac{\lambda - c(t)}{a(t)}, \qquad \nu(t) = \frac{b_0}{a(t)}. \tag{52}$$

Substituting equation (51) into (43), we obtain

$$\alpha_\lambda(t) = \alpha_\lambda(0) - \int_0^t \frac{\left[\eta^2(\tau) + \mathrm{i}\hbar\nu(\tau)\right]}{2I\hbar}\,\mathrm{d}\tau, \tag{53}$$

where

$$\eta(\tau) \equiv \mu(\tau) - n_\Phi(\tau)\hbar. \tag{54}$$

In the derivation of equation (53), we have used the following two identities:

$$\dot{\mu} = \frac{\nu\,(\mu - n_\Phi\hbar)}{I}, \qquad \dot{\nu} = \frac{\nu^2}{I}. \tag{55}$$

Here we see that in general $\alpha_\lambda(t)$ is a complex function.

Although the form of $\psi_\lambda(\theta, t) = \mathrm{e}^{\mathrm{i}\alpha_\lambda(t)}\phi_\lambda(\theta, t)$ is indeed a solution of equation (1), it does not satisfy the periodic boundary condition (see equation (12)). This problem can be resolved by defining the total wavefunction $\psi(\theta, t)$ as the summation of all $\psi_\lambda(\theta + 2n\pi, t)$ terms:

$$\begin{aligned}\psi(\theta, t) &= \sum_{n=-\infty}^{\infty} \psi_\lambda(\theta + 2n\pi, t)\\ &= \sum_{n=-\infty}^{\infty} \exp\left[\mathrm{i}\alpha_\lambda(t) + \frac{\mathrm{i}}{\hbar}\mu(t)(\theta + 2n\pi) - \frac{\mathrm{i}}{2\hbar}\nu(t)(\theta + 2n\pi)^2\right].\end{aligned} \tag{56}$$

It can also be transformed to the equivalent form below using the Poisson summation formula:

$$\psi(\theta,t)=\sqrt{\frac{\hbar}{2\pi \mathrm{i}\nu(t)}}\exp\left[\mathrm{i}\alpha_\lambda(t)+\mathrm{i}\frac{\nu(t)}{2\hbar}\theta_{\mathrm{c}}^2(t)+\mathrm{i}n_0\theta_{\mathrm{c}}(t)\right]$$
$$\times\sum_{n=-\infty}^{\infty}\exp\left[\frac{\mathrm{i}\hbar(n-n_0)^2}{2\nu(t)}+\mathrm{i}n\,(\theta-\theta_{\mathrm{c}}(t))\right]. \tag{57}$$

From these derivations it should be noted that *when using the LR method, the boundary conditions have to be carefully managed, otherwise one may get an incorrect result*.

6. A comparison of various approaches

In this section we shall show that equations (56) and (57) can be cast into the forms of equations (21) and (18), respectively. To proceed further, let us first borrow the parameters n_0 and $\theta_{\mathrm{c}}(t)$ from section 2; in combination with the results obtained in section 5, we have the simple identity

$$a(t)n_0\hbar+b(t)\theta_{\mathrm{c}}(t)+c(t)=a_0n_0\hbar+b_0\theta_0+c_0. \tag{58}$$

Comparing this result with equation (50), we find that they are very similar. For simplicity, we define

$$\lambda\equiv a(t)n_0\hbar+b(t)\theta_{\mathrm{c}}(t)+c(t); \tag{59}$$

in combination with equation (59), it is easy to obtain

$$\mu(t)=n_0\hbar+\nu(t)\theta_{\mathrm{c}}(t), \tag{60}$$
$$\eta(t)=l_{\mathrm{c}}(t)+\nu(t)\theta_{\mathrm{c}}(t). \tag{61}$$

Further, using the identity

$$\frac{\mathrm{d}\theta_{\mathrm{c}}^2(t)}{\mathrm{d}t}=\frac{2}{I}l_{\mathrm{c}}(t)\theta_{\mathrm{c}}(t) \tag{62}$$

and the identity of $\dot{\nu}$ in equation (55), we have

$$\eta^2=l_{\mathrm{c}}^2+I\frac{\mathrm{d}}{\mathrm{d}t}(\nu\theta_{\mathrm{c}}^2). \tag{63}$$

Substituting equation (63) into (53), we get

$$\mathrm{e}^{\mathrm{i}\alpha_\lambda(t)}=\frac{\mathrm{e}^{\mathrm{i}\alpha_\lambda(0)}\exp\left(-\frac{\mathrm{i}}{\hbar}\int_0^t\frac{l_{\mathrm{c}}^2(\tau)}{2I}\,\mathrm{d}\tau-\frac{\mathrm{i}\nu\theta_{\mathrm{c}}^2}{2\hbar}\right)}{\sqrt{1-\frac{\nu_0 t}{I}}}. \tag{64}$$

In addition, by defining $\mathrm{e}^{\mathrm{i}\alpha_\lambda(0)}$ and ν_0 as

$$\mathrm{e}^{\mathrm{i}\alpha_\lambda(0)}\equiv\frac{N\sqrt{\pi}}{\sigma} \tag{65}$$

and

$$\nu_0\equiv-\frac{\mathrm{i}I}{T}=-\frac{\mathrm{i}\hbar}{2\sigma^2}, \tag{66}$$

we can see clearly that equations (56) and (57) become exactly the same as equations (21) and (18). Hence, we have verified that the Fourier transform method, the time evolution method, and the Lewis–Riesenfeld method are equivalent when we choose the coefficients c_n as equation (17). This restriction is not necessary to find the two equivalent general solutions, equations (16) and (38), obtained by using Fourier expansion and time evolution methods, respectively. It turns out that the Lewis–Riesenfeld method seems to be more restrictive.

It is now interesting to discuss the physical meanings of $l_c(t)$ and $\theta_c(t)$ we have obtained. Although they originate from the classical treatment, the sense in which they play the role of dynamic variables in the corresponding classical system should be further clarified. We would like to note that in the Schrödinger picture within coordinate representation, $l_c(t)$ is the expectation value of $\hat{L} - qRA(t) = -i\hbar\partial/\partial\theta - qRA(t)$. However, $\theta_c(t)$ is not the expectation value of $\hat{\theta} = \theta$ with respect to the wavefunction obtained in equation (18); instead, it is merely the peak position of the wavepacket (see equation (21)).

In other words, the conventional Ehrenfest theorem is not directly applicable in this topologically nontrivial system. This consequence is due to the fact that we are not able to distinguish the phase between the angle θ and $\theta + 2n\pi$. Hence the $\hat{\theta}$ operator is not well defined; however $e^{i\hat{\theta}}$ is a well-defined operator, as has been demonstrated in section 4. These facts cause λ to lose its meaning as an expectation value of the $\hat{Q}$ operator.

We finally point out the relationship between the problem we have considered here and that we studied in the previous work [8]. In that work we studied the motion of a charged particle in a one-dimensional space subject to a time-varying linear potential. By doing a gauge transformation such as that mentioned in [27], the Hamiltonian in [8] can be cast into the form of equation (1); hence the two problems are equivalent if we ignore the difference of their topologies. Therefore, by ignoring the factor caused by the gauge transformation (which contains only a function of time), the circulating wavepacket solution, equation (21), can be viewed as the wavepacket solution in a one-dimensional system (see equations (24) and (40) in [8]) being folded into a ring. That is why in equation (56) the total wavefunction $\psi(\theta, t)$ can be written as the sum of all the $\psi_\lambda(\theta + 2n\pi, t)$ terms. This folding nature of the wavepacket leads to interferences between different $\psi_\lambda(\theta + 2n\pi, t)$ terms. As a result, the expectation value of the $\hat{\theta}$ operator is different from the peak position θ_c of the wavepacket.

7. Concluding remarks

In this paper, we have studied the problem of a charged particle moving in a ring subject to a time-dependent flux threading it. After analysing the problem in a classical manner, various approaches including a Fourier expansion method, a time-evolution method, and the Lewis–Riesenfeld method were considered and compared. In the Lewis–Riesenfeld approach, by appropriately managing the periodic boundary condition of the system, a time-dependent wavefunction can be obtained by using a non-Hermitian time-dependent linear invariant. The eigenfunction of the invariant can be realized as a Gaussian-type wavepacket with the peak moving along the classical angular trajectory, while the distribution of the wavepacket is determined by the ratio of the coefficient of the initial angle to that of the initial canonical angular momentum. In this circular system, we find that although the classical trajectory and angular momentum can determine the motion of the wavepacket, the peak position is no longer an expectation value of the angle operator, and the Ehrenfest theorem can not be directly applicable.

Recently, possible schemes of the experimental setup to explore the quantum dynamics of a mesoscopic ring threaded by a time-dependent magnetic flux have been proposed by either capacitively coupling the ring to a electronic reservoir [28] or applying two shaped time-delayed pulses [29]. The quantum dynamics in a time-dependent flux-driven ring should be achievable within recent fabrication capability.

Acknowledgments

The authors are grateful to D H Lin, Y M Kao, and M Moskalets for discussion of the results. This work was supported by the National Science Council, the National Central University, and the National Center for Theoretical Sciences in Taiwan.

References

[1] van Wees B J 1991 *Phys. Rev. Lett.* **66** 2033
[2] Figueira C, Faria de M, Dorr M and Sandner W 1996 *Phys. Rev.* A **55** 3961
[3] Tang C S and Chu C S 1999 *Phys. Rev.* B **60** 1830
[4] Liu M T and Chu C S 2000 *Phys. Rev.* B **61** 7645
[5] Guedes I 2001 *Phys. Rev.* A **63** 034102
[6] Bekkar H, Benamira F and Maamache M 2003 *Phys. Rev.* A **68** 0160101
[7] Maeda H and Gallagher T F 2004 *Phys. Rev. Lett.* **92** 133004
[8] Luan P G and Tang C S 2005 *Phys. Rev.* A **71** 014101
[9] Aharonov Y and Bohm D 1959 *Phys. Rev.* **115** 485
[10] Aronov A G and Sharvin Yu V 1987 *Rev. Mod. Phys.* **59** 755
[11] Chakraborty T and Pietiläinen P 1994 *Phys. Rev.* B **50** 8460
[12] Berry M V 1984 *Proc. R. Soc.* A **392** 45
[13] Aharonov Y and Anandan J 1987 *Phys. Rev. Lett.* **58** 1593
[14] Landauer R and Büttiker M 1985 *Phys. Rev. Lett.* **54** 2049
[15] Gefen Y and Thouless D J 1987 *Phys. Rev. Lett.* **59** 1752
[16] Mullen K, Ben-Jacob E, Gefen Y and Schuss Z 1989 *Phys. Rev. Lett.* **62** 2543
[17] Büttiker M, Imry Y and Landauer R 1983 *Phys. Lett.* A **96** 365
[18] Chandrasekhar V *et al* 1991 *Phys. Rev. Lett.* **67** 3578
[19] Mailly D, Chapelier C and Benoit M 1993 *Phys. Rev. Lett.* **70** 2020
[20] Morpurgo A F *et al* 1998 *Phys. Rev. Lett.* **80** 1050
[21] Schuster R *et al* 1997 *Nature* **385** 417
[22] Bayer M, Korkusinski M, Hawrylak P, Gutbrod T, Michel M and Forchel A 2003 *Phys. Rev. Lett.* **90** 186801
[23] Levy L P, Dolan G, Dunsmuir J and Bouchiat H 1990 *Phys. Rev. Lett.* **64** 2074
[24] Stern A 1992 *Phys. Rev. Lett.* **68** 1022
[25] Lewis H R Jr 1967 *Phys. Rev. Lett.* **18** 510
[26] Lewis H R Jr and Riesenfeld W B 1969 *J. Math. Phys.* **10** 1458
[27] Bauer J 2002 *Phys. Rev.* A **65** 036101
[28] Arrachea L 2002 *Phys. Rev.* B **66** 045315
[29] Matos-Abiague A and Berkdar J 2005 *Phys. Rev. Lett.* **94** 166801